Pascal and Leibniz: Sines, Circles, and Transmutations

The title of the last section promised some discussion of sine curves. In Roberval's construction a sine curve was drawn and used to find the area of the cycloidal arch (see Figure 2.13b), but Roberval called this curve the "companion of the cycloid." He did not see this curve as a graph of a sine or cosine, and neither did any of his contemporaries. They did discuss sines and cosines, however, and it is important to understand the conceptual point of view that was then standard. Before I present my last example of curve drawing from Leibniz, I must also return to some of the issues raised in section 2.3. Leibniz's use of the characteristic triangle (Figure 2.3b) was directly inspired by the work of Pascal concerning sines.

Trigonometric functions were not defined using ratios, or the unit circle, until the textbooks of Euler were published in 1748 (Euler, 1988). Both Ptolemy and Arabic astronomers made detailed tables of the lengths of chords subtended by circular arcs (Katz, 1993). That is, given two points, A and B, on a circle, to find the length of the line segment $\overline{AB}$. Such tables were usually made for a circle of a given (large) radius, and then scaled for use in other settings. It was also found useful by Arabic astronomers to have tables of half chords, and such tables became known in Latin as tables of "sines." [1] Since the perpendicular bisector of any chord passes through the center of the circle, half chords on the unit circle are the same as our sines, but I want to stress that trigonometric quantities were seen as <u>lengths</u>. Tangents were seen as lengths marked on a tangent line. Secants were the lengths from the center of the circle to the tangent line. Etc. In Figure 2.14a, PB is a sine, CD is a tangent, and AD is a secant, regardless of what parameter is used to index these lengths.

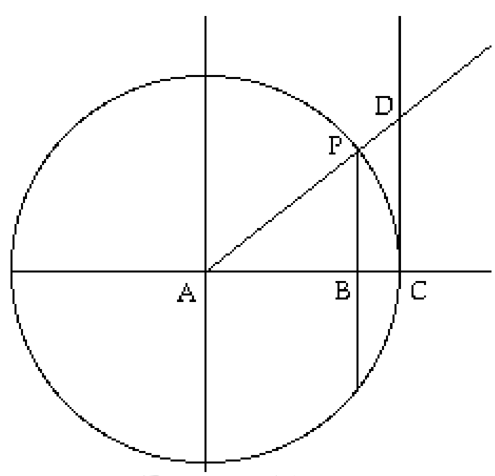

Figure 2.14a

[1] The Arabic word for "half chord" closely resembled the Arabic word for a bay of water. Early Latin translators confused the two and translated the Arabic "half chord" into the Latin word "sinus" meaning bay or cavity.

The parameter from which these trigonometric lengths were indexed was implied by usage, but not always stated explicitly. Some form of systematic equal division was used to make any particular table. As analytic geometry evolved, parameters had to be stated more clearly. For example, in Figure 2.13a, is PB a "sine" or an "ordinate" to the circle with respect to the axis $\overleftrightarrow{AC}$? Pascal and Roberval were quite clear on this point. If the line $\overleftrightarrow{AC}$ is divided into equal increments, and then perpendiculars are erected to the circle, then those segments are "ordinates." On the other hand, if the circle is divided into equal pieces of arc length, and then perpendicular segments are dropped to the axis, then those segments are "sines" (Struik, 1969). See Figure 2.14b, where on the left the diameter is divided into sixteen equal segments, and on the right the half circle is divided into sixteen equal arc lengths. Pascal did not restrict this terminology to the circle, but used it for any curve .

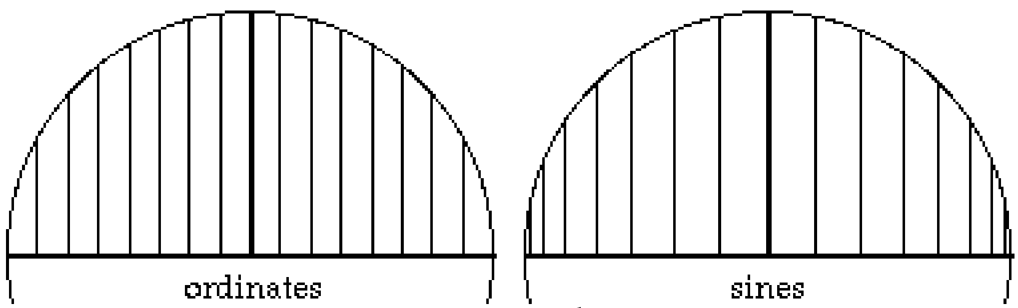

Figure 2.14b

In any given situation, the size of the increment was determined by the nature of the problem at hand, and the level of accuracy that was desired or possible. The smallest increment being used was most often taken as the unit of measurement for a problem. In the early work of Leibniz, $dx = 1$ represents these small increments, and there is no distinction between dy and dy/dx. Leibniz did not distinguish between dx and Δx; he used only dx, and it meant simply the difference of the quantity x. The ratio dy/dx entered into his later work, so that he could discuss rates using units of measurement other than the smallest increment dx (Child, 1920).

In 1659 Pascal published his work "On the sines of a quadrant of a circle" in which he established a series of propositions which are algebraically (but not conceptually) equivalent to integrating all of the integer powers of the sine function (Struik, 1969). I will describe only his first example. Figure 2.14c shows an increment of circular arc length DF, together with a segment $\overline{QR}$ tangent at the midpoint P. $\overline{CP}$ and $\overline{CA}$ are radii of the circle, and $\overline{QK}$, $\overline{PB}$, and $\overline{RL}$ are all perpendicular to $\overline{CA}$. Since $\overline{QR}$ is perpendicular to $\overline{PC}$, triangles ΔERQ and ΔBPC are similar. Hence $PB \cdot QR = ER \cdot CP$

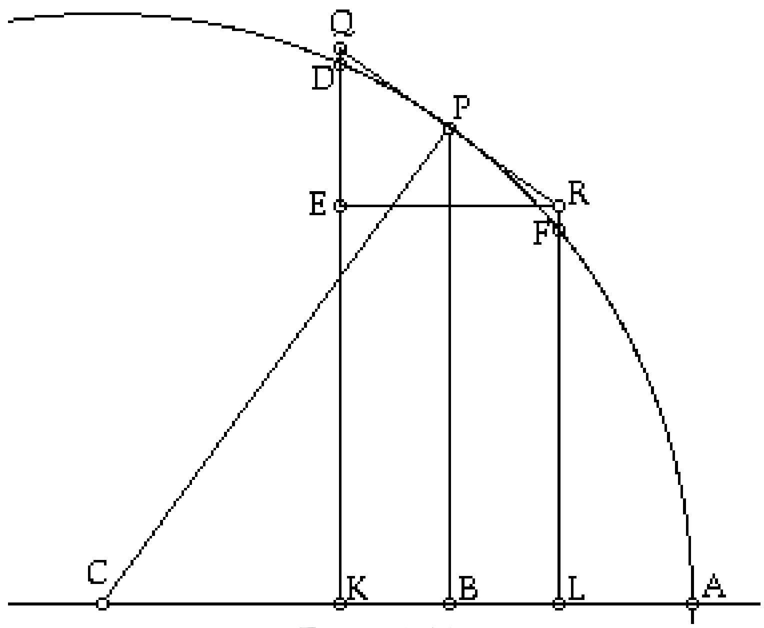

Figure 2.14c

Imagine any arc (≤ 90°) of a circle divided up into equal pieces of arc length such that it is impossible to distinguish those arc length increments from the corresponding tangent line segments. Even in Figure 2.14c they are very close ($QR = 1.40$ in.; arc $DF = 1.44$ in.). Using his arc length increment QR as his unit, Pascal then concluded, from the similarity statement above, that if one sums up the sines of the increments ($PB = PB \cdot QR$), then the sum is always equal to the portion of the horizontal axis between the first and last of the sines, multiplied by the radius of the circle; i.e. the sum of the changes ER in the cosine, times the radius CP (Struik, 1969).

One could transform this statement in the later notation of Leibniz (i.e. modern calculus) by a change of unit. Using the radius of the circle CP as a unit (rather than the arc length increment QR), and using θ as an arc length parameter (i.e. radians), then $QR = d\theta$, $ER = KL = d(\cos\theta)$, and the similarity statement $PB \cdot QR = ER \cdot CP$ becomes $\sin\theta \cdot d\theta = d(\cos\theta)$. Summing this one obtains: $\int_a^b \sin\theta \cdot d\theta = \cos a - \cos b$. Leibniz created the modern integral symbol as an "s" for summation. Just as he did not separate the concepts of dx and Δx, he did not separate integration ($\int$) from summation ($\sum$). Thinking geometrically, the issue of minus signs does not arise, because ER (the change in cosine) is a geometric line segment. The arc can be measured either clockwise or counterclockwise.

Pascal made a whole series of increasingly complex arguments like the one above. Transformed into Leibnizian notation, they amount to a series of integrations for integer powers of the sine, solved by changes of variable. In the work of Pascal, however, all of these changes of variable were done geometrically, using similarity and projection. Two themes from this work deeply affected Leibniz, and became central in his thinking (Child, 1920). First, the small characteristic triangles along a curve can be analyzed by finding large ones which are similar to them (see Section 2.3). In the case just mentioned, triangle ΔERQ is the characteristic triangle which is similar to the large triangle ΔBPC. Second, there are useful connections between tangents and areas that can be exploited through the finding of such similar triangles.

A few years before his death in 1716, Leibniz wrote "The history of the origins of differential calculus" (Child, 1920). This essay centered on two themes. First, that his notation of differences and summations was developed from his study of tables of numbers, and the patterns that he found there. Second, that this notation from tables could be consistently applied to geometry, and was capable of yielding all known results concerning areas, volumes, tangents, and arc lengths. Leibniz described his original insights into the consistency between geometry, and his new algebraic notation, by focusing on what came to be know as the "transmutation of curves" which involved a particular example of a large triangle which is similar to the characteristic one. He used this triangle as a way to draw new curves from existing ones. This method of curve drawing produces, from the original curve, a new "transmuted curve" which bounds areas that are closely related to the areas bounded by the original curve. As we shall see, this transmutation is closely akin to integration by parts. This technique was first investigated by Leibniz early in his career, in 1673, and was described in his letters to Newton (Turnbull, 1960; Child, 1920; Edwards, 1979).

In order to understand the curves and derivations of Leibniz, I have constructed the following three point rectilinear example, as an introduction. This example is not from the writings of Leibniz, but it I think it could help students understand his conceptual approach. Imagine a piece-wise linear "curve" passing through the three points A, B, and C, where A is the origin (see Figure 2.14d). If all three points are collinear then the area under this curve is the area of triangle ΔACG, which is equal to $\frac{1}{2}x'y'$, where x' and y' are the coordinates of C.[2] If B is moved up off the line $\overleftrightarrow{AC}$ then the area under the curve ABC equals $\frac{1}{2}x'y' + \text{Area}(\Delta ABC)$. Leibniz's transmutation technique shows us how to find the area of ΔABC, by looking at a new curve which is drawn by monitoring where the tangents ($\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ in this case) to the original curve intersect the y-axis.

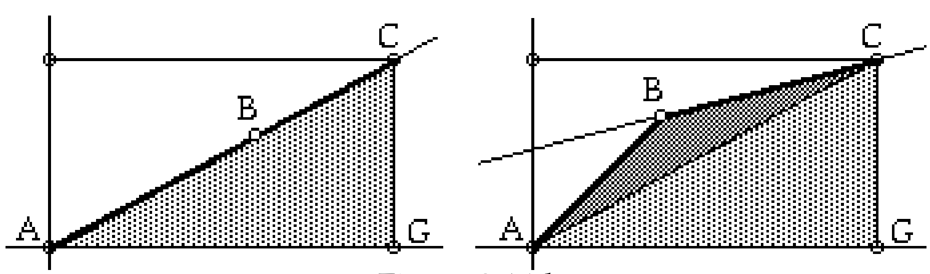

Figure 2.14d

If we let the point U be the intersection of line $\overleftrightarrow{BC}$ (tangent) with the y-axis, then Figure 2.14e shows how to construct a right triangle ΔDFG, which is equal in area to triangle ΔABC, by making $FG = AU$, and $DG = BH$. To see this area equality, first

[2] Throughout this section prime notation such as x' and y' will be used to denote fixed endpoint values of variables, and has no relation to derivatives. Any mention of derivatives will use strictly Leibniz notation.

construct $\overline{AN}$ perpendicular to $\overline{BC}$. Now triangles ΔBHC and ΔANU are similar. This triangle ΔANU is an example of a triangle which is similar to the characteristic triangle at the point B. Letting $AU = z$ and $AN = p$, from the similarity one sees that $z \cdot dx = p \cdot ds$.

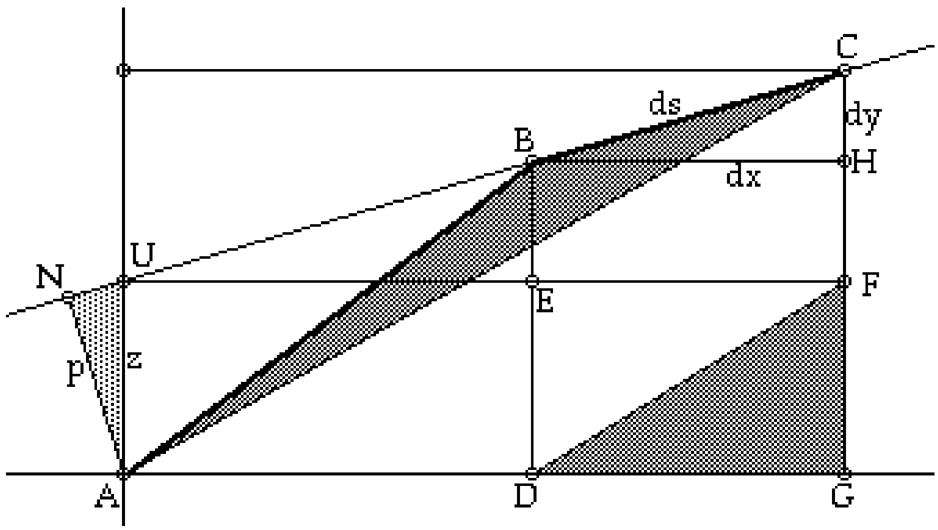

Figure 2.14e

Since triangle ΔABC can be seen as having a base of ds and height of p, its area is $\frac{1}{2} p \cdot ds$, while triangle ΔDFG has a base of dx and a height of z and hence an area of $\frac{1}{2} z \cdot dx$. Hence the similarity tells us that the two darkly shaded triangular areas are equal. This construction can be seen to be helpful in the sense that triangle ΔABC has been "transmuted" into the right triangle ΔDGF, which sits nicely in the coordinate system. If $C = (x', y')$, then the area under the curve ABC equals $\text{Area}(\Delta AGC) + \text{Area}(\Delta ABC) = \frac{1}{2} x'y' + \text{Area}(\Delta DGF)$. Using *Geometer's Sketchpad*, one can drag point B to different positions and watch the changes in triangle ΔDFG, while monitoring the areas with meters.

In order to generalize this construction to non linear curves, Leibniz defined a new "transmuted curve" constructed from any given curve, relative to an axis of ordinates, as follows:

Definition: Given any curve and a system of perpendicular abscissas and ordinates, for each point (x, y) on the given curve define a point (x, z) on the transmutation curve as the one which has the same abscissa x, and has, as its ordinate, the length z, where z is equal to the y-intercept of the tangent line to the original curve at the point (x, y).

In Figure 2.14e, starting with the curve ABC, this new transmuted curve would be the piecewise linear path through $ADEF$ (i.e. a step function in the modern sense). Leibniz then expressed the area under the original curve from A to C as: $\frac{1}{2} x'y' + \frac{1}{2}(\text{area under the transmuted curve})$, where $A = (0,0)$ and $C = (x', y')$. The first term is the area of triangle ΔACG, and second term is equal to the "extra area" under the curve contributed by triangle ΔABC.

This general curve drawing technique can applied to any curve where one can construct the tangent lines at all points and thus monitor their intersections with the axis of ordinates. The area under the new transmuted curve, drawn from the original, will then be used to find the area under the original curve. This geometrical construction yielded, for Leibniz, the area formula above, which is algebraically (but not conceptually) equivalent to the technique now known as "integration by parts." Leibniz, however, developed his transmutation of curves prior to his algebraic notations, such as the product rule. He developed a general algebraic language (i.e. the calculus) only after he had investigated many examples of his geometric transmutation, and had seen the generality of the technique. The extension of his language and notation to geometry grew from his experience with curve generation and transmutation.

I will next apply this curve drawing technique to the circle, because it was Leibniz's favorite example (Child, 1920). In doing so, I will also derive again the transmutation area formula in a more general setting using the mature notation of Leibniz, although this derivation is essentially the same as the one from Figure 2.14e. This general transmutation formula (2.14-1) which I will derive is then valid for any curve with tangent lines.

I start by letting the point P rotate around a circle while dragging its tangent line $\overleftrightarrow{PT}$ with it (see Figure 2.14f). Letting the diameter $\overline{AB}$ be the axis of abscissas measured from A (i.e. $A = (0,0)$), at each position of P, let $\overleftrightarrow{AN}$ be a perpendicular from A to the tangent line. The triangle ΔAUP (shaded) is then similar to the small characteristic triangle along the curve at P. Construct a new curve by tracing the locus of the point R, where R and P always have the same abscissa AS, and the new ordinate RS is always equal to AU (the y-intercept of the tangent line at P). This new curve is drawn from the circle in Figure 2.14f, using an animation in *Geometer's Sketchpad*. See Figures 2.14i and 2.14j to see this same technique applied to the cycloid and the hyperbola .

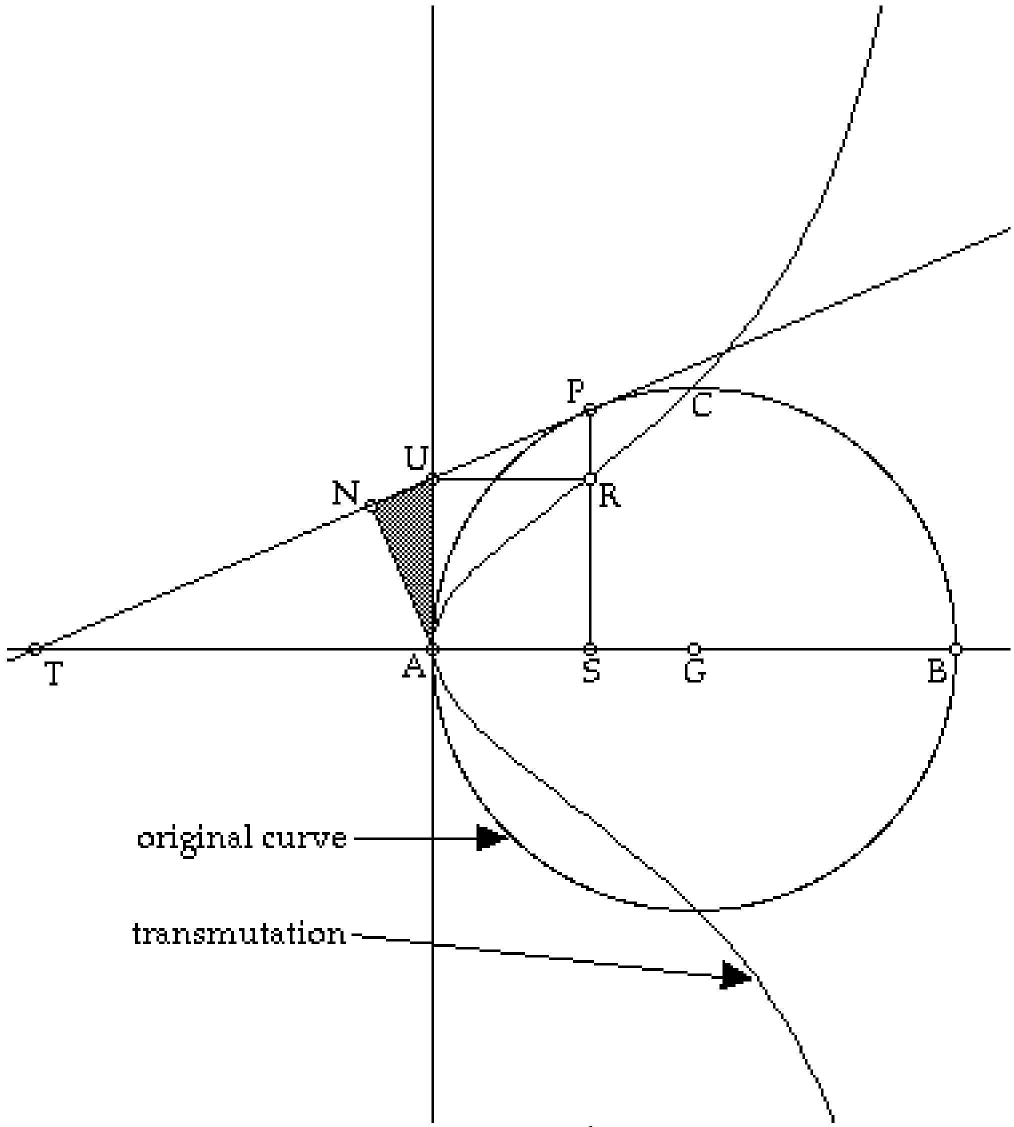

Figure 2.14f

Using the mature notation of Leibniz, his general transmutation area formula can be derived as follows. As before, let $x = AS$ and $y = SP$ be the abscissas and ordinates of the original curve, and let $z = AU$ be the ordinates of the transmuted curve (i.e. $P = (x,y)$ and $R = (x,z)$). Since z is the intercept of the tangent line to the original curve, $z = y - x \cdot \frac{dy}{dx}$. Now let $p = AN$. Since triangle ΔANU is similar to the characteristic triangle at the point P (sides dx, dy, and ds) we have: $\frac{dx}{ds} = \frac{p}{z}$, where s is arc length along the original curve. Hence $p \cdot ds = z \cdot dx$.

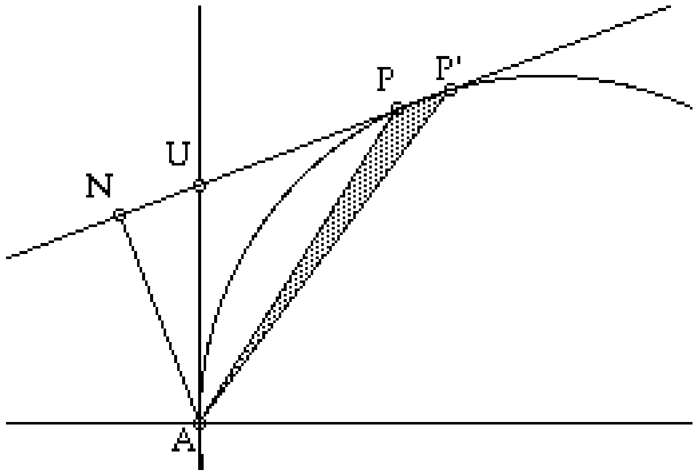

Figure 2.14g

Now imagine two points, P and P', on the original curve that are so close together that the line $\overleftrightarrow{PP'}$ is essentially the tangent line at P (see Figure 2.14g). Now imagine the slender triangle $\Delta APP'$ (shaded). Thinking of $PP' = ds$ as the base this triangle, it has a height of $AN = p$, and so its area is

$\frac{1}{2} \cdot p \cdot ds = \frac{1}{2} \cdot z \cdot dx$ (from the above similarity argument).

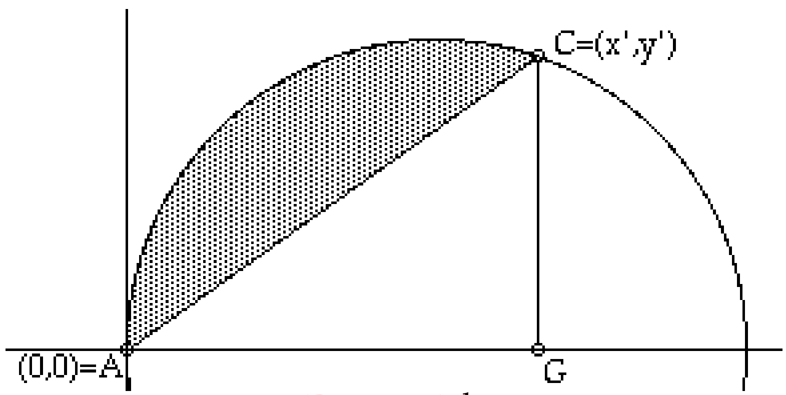

Figure 2.14h

Now choose any fixed point $C = (x', y')$ on the original curve (see Figure 2.14h). The area of the shaded sector AC is the sum of the slender triangular sectors in Figure 2.14g, which equals $\frac{1}{2}\int_0^{x'} p \cdot ds = \frac{1}{2}\int_0^{x'} z \cdot dx$. The area under the curve between A and C is equal to the area of triangle ΔAGC plus the area of the shaded sector, which is to say:

$$\text{(2.14-1)} \qquad \int_0^{x'} y \cdot dx = \frac{1}{2}(x' \cdot y') + \frac{1}{2}\int_0^{x'} z \cdot dx$$

This is Leibniz's transmutation formula. As I showed in the three point example, it says that the area under any curve is the triangle ΔAGC plus half the area under the transmutation curve. If one substitutes for z from the original definition $z = y - x \cdot \frac{dy}{dx}$, and then solves for the integral of y, then the statement becomes the usual integration by parts formula written for definite integrals: $\int_0^{x'} y \cdot dx = [xy]_{(0,0)}^{(x',y')} - \int_0^{y'} x \cdot dy$.

I make this last statement only for modern readers to see the connection with the usual algebraic approach to "integration by parts" which is taught as an application of the product rule written backwards. The formula which Leibniz applied with great success, in a large number of examples, is the geometric statement (2.14-1) which expresses the area under the original curve in terms of the triangle ΔAGC plus half the area under the new transmutation curve. This transmutation curve can be drawn by a simple linkage, provided that the method which drew the original curve included a construction of tangent lines. This chapter has already shown that that can be done for a great variety of curves. This technique of Leibniz is a response to experience with the tools of the day. The tools mediated the knowledge.

Leibniz first stated his formula in a purely geometric language. For him, it was a major breakthrough in seeing how tangents could be used to determine areas. He developed this method just before he constructed his first notations for calculus, which would later include such statements as the "product rule," the "quotient rule," and the "chain rule." Integration by parts is usually introduced to students as a purely algebraic manipulation of the product rule, but for Leibniz, such a derivation was an algebraic confirmation of the geometric experiments that he carried out using his transmutation technique.

This transmutation of curves can be used in a variety of ways to conduct the kind of critical experiment that tests the validity of a new language against independently established results. Here is how Leibniz applied the transmutation formula to obtain an expression for the area under any part of circle, and then tested his expression by using it to compute the area of a quarter of a circle of radius one. In Figure 2.14f, let G be the center of the circle of radius one: $AG = 1$, and $A = (0,0)$. The equation of the upper half of the circle is $y = \sqrt{2x - x^2}$, and since the tangent is always perpendicular to the radius, $\frac{dy}{dx} = \frac{1-x}{y}$. Hence the equation of the newly drawn, bell shaped, transmutation curve is:

$$z = y - x\frac{1-x}{y} = \sqrt{\frac{x}{2-x}} \quad \text{or} \quad x = \frac{2z^2}{1+z^2}$$

By synthetic division, $x = 2\{z^2 - z^4 + z^6 - z^8 + \cdots\}$

Now the area under the circle between $A = (0,0)$ and any fixed point (x', y') is given by (2.14-1) as:

$$\int_0^{x'} y \cdot dx = \frac{1}{2}(x' \cdot y') + \frac{1}{2}\int_0^{x'} z \cdot dx$$

Since z, in terms of x, also involves a square root, Leibniz rewrote the area under the transmutation curve by subtracting its complement from the rectangle containing it. Hence, as an integral in x subtracted from the circumscribed rectangle with area $x' \cdot z'$ (where z' is the value of z at $x = x'$), Leibniz obtained:

$$\int_0^{x'} z \cdot dx = (x' \cdot z') - \int_0^{z'} x \cdot dz$$

This new integral can now be expressed using the synthetic division above and then integrated term by term.[3] Hence the area under the circle is:

$$\begin{aligned}\int_0^{x'} y \cdot dx &= \frac{1}{2}(x' \cdot y') + \frac{1}{2}(x' \cdot z') - \frac{1}{2}\int_0^{z'} 2(z^2 - z^4 + z^6 - z^8 + \ldots)dz \\ &= \frac{1}{2}(x' \cdot y') + \frac{1}{2}(x' \cdot z') - \left[\frac{1}{3}z^3 - \frac{1}{5}z^5 + \frac{1}{7}z^7 - \frac{1}{9}z^9 + \cdots\right]_0^{z'} \\ &= \frac{1}{2}(x' \cdot y') + \frac{1}{2}(x' \cdot z') - \left(\frac{1}{3}z'^3 - \frac{1}{5}z'^5 + \frac{1}{7}z'^7 - \frac{1}{9}z'^9 + \cdots\right)\end{aligned}$$

In order to test the validity of this expression against known results in geometry, Leibniz checked it on the area of a quarter circle where $x' = y' = z' = 1$. The expression above then asserts that:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Checking this series empirically one finds that it does indeed converge to $\frac{\pi}{4}$, although very slowly.[4] This alternating odd harmonic series provided a new way to compute π, and it was the first known expression for π as a sum of rational numbers. Far more important for Leibniz was the fact that this calculation was a confirming experiment that showed that his new language was indeed a tool which was consistent with the long established traditions of geometry. This showed him that the notation that he developed from tables could be consistently applied to geometry.

The confirmation of a new mathematical approach using as a check a new computation of π was a favorite procedure during the seventeenth century. In 1655,

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[3] The calculation of areas and volumes that involve only the summation (integration) of polynomials had been known since the tenth century from the work of the Arabic mathematician, al-Haitham. In the early seventeenth century the Arabic work on polynomial summations had been extensively elaborated by Cavalieri, Fermat, Pascal and Wallis (Struik, 1969; Dennis & Confrey 1993).

[4] Many terms of this series are needed to obtain even a minimal level of accuracy. As a practical calculation of π, this method is abysmal. More effective methods for evaluating π had been known since ancient times (Archimedes, 1952), such as the doubling on the number of sides in regular polygons, which gives π as a nested series of square roots. This was not a new practical calculation; it was a confirming experiment for the validity of a new language.

A modern calculus book would evaluate the area under a section of the circle by making a trigonometric substitution into an integral. Such an approach looks simple, but is only useful if one already has a means to evaluate trig functions. Such means usually involve an infinite series. The series usually used for computing trig functions were first derived from series which expressed areas, like the one discussed here. (See Newton, 1967; Dennis & Confrey, 1993; Edwards, 1979).

John Wallis had used exactly this method to confirm his assertions about the consistency of algebra and geometry in his *Arithmetica Infinitorum* (1972). He had justified his extensive use of table interpolations by showing that they implied a value for π, given as an infinite product, which was consistent with other computations from geometric constructions, such as the iterated square root procedure of Archimedes (Dennis & Confrey, 1993). This work of Wallis had a profound effect on the work of both Leibniz and Newton (Newton,1967; Child, 1920).

The use of synthetic division to create an infinite series was a technique pioneered by Nicolaus Mercator (1620 -1687), and used extensively by Leibniz, Newton and other mathematicians of the period. Leibniz saw his transmutation technique as a general method for finding rational expressions that could replace those integrands which contained root extractions, thereby creating series expressions for those areas using, first, synthetic division, followed by term by term polynomial integration (i.e. what Leibniz called "Mercator's method") (Turnbull, 1960; Boyer, 1968). In the example above, for the circle, it should be noted that the transmutation curve for the circle is a cubic curve, since it contains an xz^2 term. In general the transmutation curves for conic sections are of third or fourth degree, or what Descartes called "curves of the second class."

Leibniz drew remarkably few figures in his descriptions of his work. He tended to prefer elegant tabular displays of the coefficients that occurred in his series expansions (see for example his August, 1676 letter to Newton in Turnbull, 1960, pp. 65 -71). These tabular displays reveal some remarkable connections between trigonometric series like the one above and logarithmic series which come from hyperbolic areas via transmutation (see figure 2.14j). Despite the paucity of figures in the original discussions of Leibniz, *Geometer's Sketchpad* allows one to create a remarkable set of curves based on the transmutation construction. It is these sets of curves with their corresponding equivalent areas which I feel could provide fertile ground for student investigations.

A simpler and interesting special case comes from transmuting a parabola using the line tangent to the vertex as the axis of ordinates (this example does not appear in Leibniz). I invite the reader to make his/her own figure (see Section 2.4). One could then choose a coordinate system so that the parabola has the equation: $y = \sqrt{x}$. From the geometric properties of parabolic tangents discussed in Section 2.4 (i.e. the subtangent is always twice the abscissa), it can be seen that the transmutation curve of this parabola is another parabola whose ordinates are half those of the original curve, i.e. $z = \frac{\sqrt{x}}{2}$. If one considers the area under the curve from (0,0) to (x',y') transmutation tells us that: $\int_0^{x'} \sqrt{x}\,dx = \frac{x'y'}{2} + \frac{1}{2}\int_0^{x'} \frac{\sqrt{x}}{2}\,dx$.

Since the same integral appears on both sides one can solve for it to obtain: $\int_0^{x'} \sqrt{x}\,dx = \frac{2}{3}x'y' = \frac{2}{3}y'^3$. By looking at the complement of the rectangle with sides x' and y', this is equivalent to the usual integration of the parabola written as $\int_0^{y'} y^2 dy = \frac{1}{3}y'^3$. Note that this does not involve any use of "anti-derivatives," but is instead a linguistic coding of purely geometric properties that built the parabola.

I wish to return to the cycloid and discuss another of Leibniz's application of the transmutation technique (Edwards, 1979). Roberval's argument (Section 2.13) showed that the area of an entire cycloidal arch was three times the area of the circle that generated it, but this argument depended on the symmetry of the companion (sine) curve, and thus did not yield the values of arbitrary sections of cycloidal area. Using the constructed tangents one can draw the transmutation curve, and arrive at this general result. Figure 2.14i shows one half of a cycloidal arch drawn sideways (i.e. the wheel is rolling along the vertical line $\overleftrightarrow{FD}$). It turns out that the transmutation curve traced by the point R is exactly the same as Roberval's companion curve, i.e. a sine-shaped curve. This follows from the fact that the tangent at P is parallel to $\overline{AB}$, and so $PB = AU$, and hence by subtraction $PR = SB$.

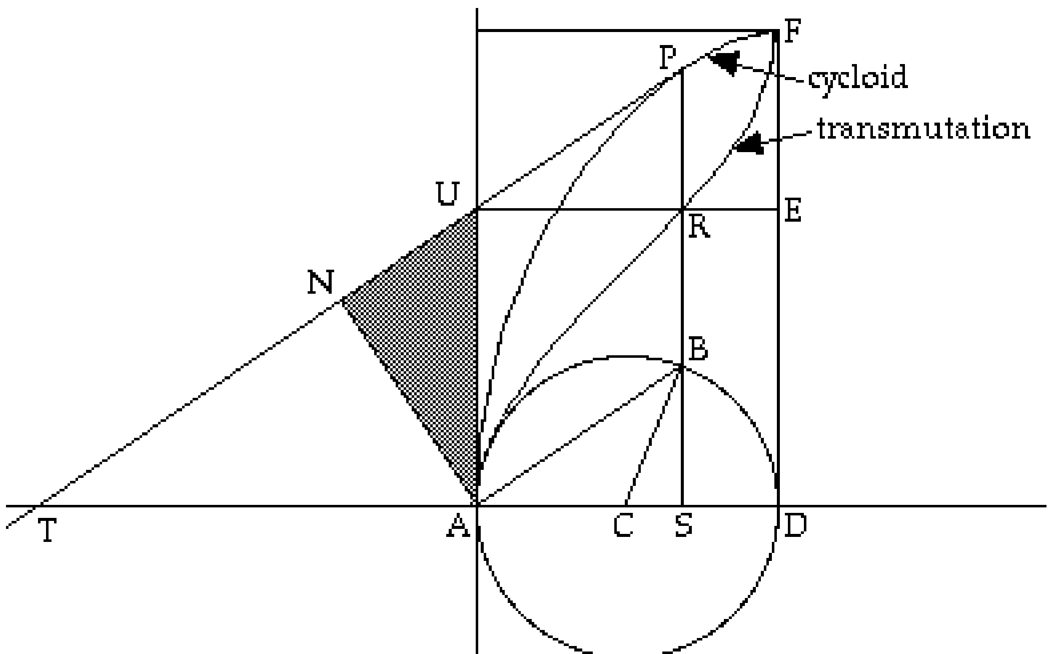

Figure 2.14i

Leibniz determined the area under the cycloid over any portion of the axis $\overleftrightarrow{AD}$ as follows. Introducing variables as before; let the abscissas x be measured from A along $\overleftrightarrow{AD}$, and let y, z, and w be the ordinates, respectively, of the three curves shown in the figure, those being the cycloid, sine, and circle. That is to say let: $P = (x,y)$, $R = (x,z)$, and $B = (x,w)$. Since $w = SB = PR$, then $y = z + w$. From the motion that produced the cycloid one knows that the circular arc length from B to D will equal FE, while the circular arc length from A to B will equal $ED = AU = PB = RS = z$. This says that the locus of R can be seen as a graph of arc length against height on the circle, which makes the curve sine shaped.

Let the constant radius of the generating circle be $r = AC$. Suppose one seeks the area under the cycloid between 0 and some x'. Let y', w', and z' be the corresponding endpoint values. The general transmutation formula (2.14-1) says that:

$$\int_0^{x'} y \cdot dx = \frac{1}{2}\left(x' \cdot y' + \int_0^{x'} z \cdot dx \right)$$
$$\int_0^{x'} (z + w) \cdot dx = \frac{1}{2}\left(x' \cdot y' + \int_0^{x'} z \cdot dx \right)$$

Solving this for $\int_0^{x'} z \cdot dx$ yields: $\int_0^{x'} z \cdot dx = x' \cdot y' - 2\int_0^{x'} w \cdot dx$

Since w represents the ordinates along the circle, $\int_0^{x'} w \cdot dx$ is the area under the circle from 0 to x', which can be expressed as the area of the circular sector ABC plus (or minus) the area of the triangle ΔCBS (see figure 2.14i). In my figure $x' > r$, but if $x' < r$ then one subtracts the triangle from the sector, in which case $x' - r$ will be negative. Using $z' = (\text{arclength from } A \text{ to } B)$ to express the area of the sector one obtains:

$\int_0^{x'} w \cdot dx = \frac{1}{2} r z' + \frac{1}{2} w'(x' - r)$.

Now

$$\int_0^{x'} z \cdot dx = x'y' - 2\left(\frac{1}{2} r\, z' + \frac{1}{2} w'(x' - r) \right) = x'(z' + w') - rz' - w'(x' - r)$$
$$= rw' - z'(r - x')$$

Substituting this back into the original transmutation formula yields:

$$(2.14\text{-}2) \qquad \int_0^{x'} y \cdot dx = \frac{1}{2} x'y' + \frac{1}{2}(rw' - z'(r - x'))$$

which gives the area under a cycloidal section in terms of the radius of the generating circle r, the endpoint $P = (x', y')$, the arc length of rotation z' and the ordinate to the circle w'. These are the constants which are natural to the action of drawing the curve. If one attempts to use the transmutation formula in a strictly algebraic sense then $\int_0^{x'} z \cdot dx$ calls for an integral of the arcsine function. This is not the conceptual approach taken here. In order to apply formula (2.14-2) one does need to know the arc length z' but the action which drew the cycloid produced line segments which were equal to this arc length. Although this approach is quite analytic one should note that the parametric equations of the cycloid were never written down or directly used. Only later did Leibniz write down the parametric equations and then use them to further test his calculus notations.

If one applies Formula 2.14-2 to one half of the cycloidal arch then $x' = 2r,\ y' = \pi r,\ w' = 0,\ z' = \pi r$. Using these values the formula yields $\frac{3}{2}\pi r^2$. This is in accordance with Galileo's experiments, and Roberval's geometry. Pascal had earlier given a general geometric solution for the area of a cycloidal section against which Leibniz's formula (2.14-2) could also be checked. It is this checking back and forth between geometry and the new language of calculus that gave people faith in the new linguistic constructions of Leibniz.

Leibniz applied his transmutation technique to the hyperbola and obtained an infinite series for the area between a hyperbola and its axis of symmetry, in much the same way that he found the circular areas. This series intrigued him because of its relations to other series, found by Newton and Mercator, for the calculation of areas between hyperbolas and their asymptotes (called by Mercator "natural logarithms") (Turnbull, 1960; Edwards, 1979). I will not describe all of these calculations, but I will draw the transmutation curve because most historical accounts provide either no figures or highly distorted ones. Using the envelope construction from Section 2.6 to draw the hyperbola with its tangents, one can trace the locus of the transmutation curve using one vertex A as the origin. Figure 2.14j shows all three branches of this curve for the hyperbola with vertices A and B. This curve has a cubic equation that is very similar in form (a single sign change) to the one for the circle, but the appearance of the curve is quite different. In this case, the equation of the hyperbola is $y = \sqrt{2x + x^2}$, and the equation of the transmutation curve is $x = \dfrac{2z^2}{1-z^2}$. One then proceeds to construct an infinite series, as in the circular case, via Mercator's method and the transmutation area formula (Turnbull, 1960; Edwards, 1979).

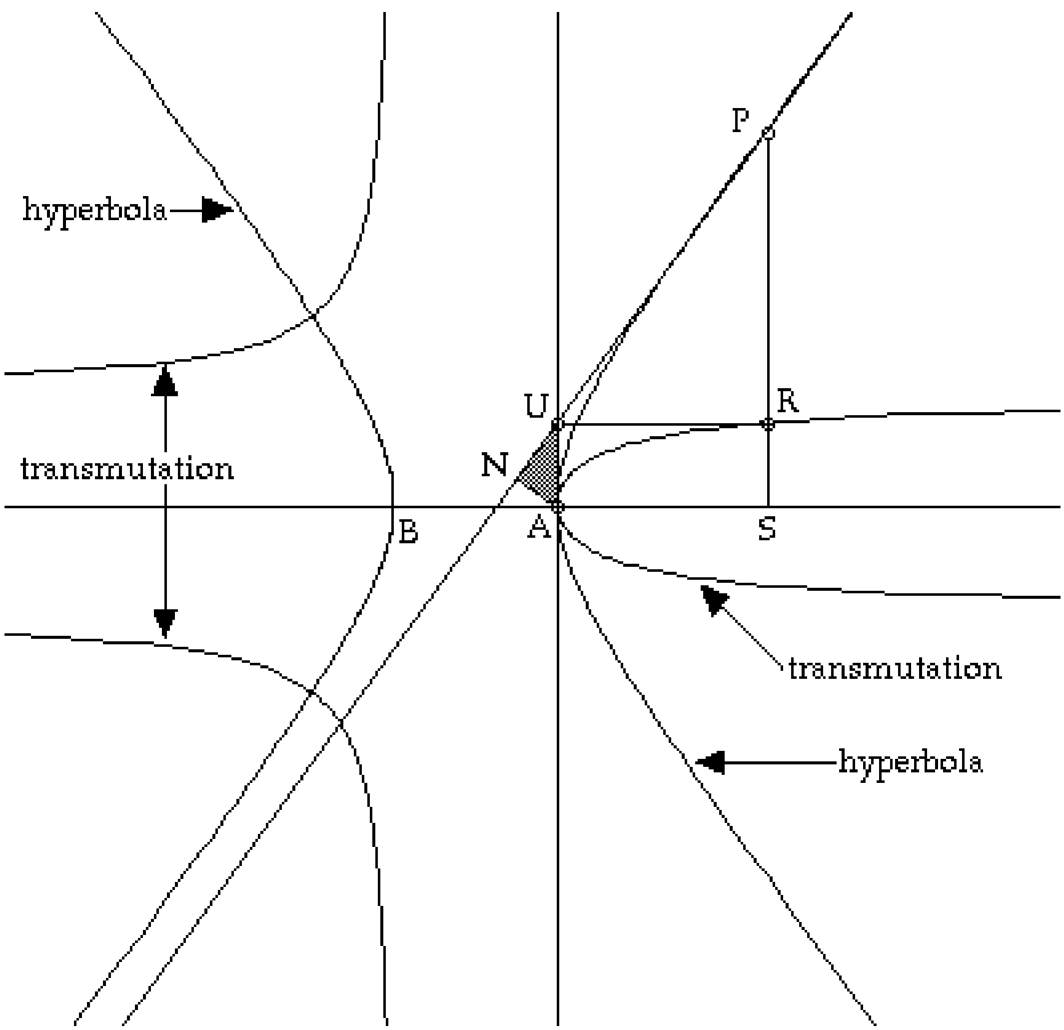

Figure 2.14j

What I find educationally valuable is the experience of accurately drawing these curves without any reference to equations or even the establishment of a scale. Using the previous conic drawing methods, one can draw a large variety of transmutation curves, depending on where one chooses to place the line on which the tangent intercepts z are to be observed. The examples chosen by Leibniz had a specific algebraic purpose, which was to rewrite areas in a rational form, so that synthetic division and term by term integration would yield an infinite series, but I find these curves interesting in their own right. They provide an iterative way to generate curves of increasing degree whose areas have a special relation to the original curve from which they were drawn.

Choosing different lines on which to intersect the tangents gives a fascinating geometric view of all possible different ways to integrate by parts. Not all of these choices will yield an algebraic simplification. For example if one constructs a transmutation curve for the hyperbola $y = 1/x$ by using the y-axis (i.e. an asymptote) for the construction then the transmutation curve is another hyperbola with the same asymptotes which is just a multiple of the original curve. This is equivalent to an application of integration by parts which is algebraically circular.

Experimenting along these lines can produce beautiful and fascinating new curves. Figure 2.14k shows another hyperbolic transmutation curve drawn with respect to a line at a skewed angle to the axis of symmetry of the hyperbola. This time the transmutation curve has two branches instead of three. Leibniz's area relation still holds. That is, the difference between the area under the hyperbola (traced by P) and the triangle ΔAPG is always one half of the area under the curve traced by R (from A to any given x). Looking at this figure and the previous one, a variety of graphic observations could emerge for discussion. For example, what does it mean when the original curve and the transmutation curve cross each other? When the original curve becomes quite straight near its asymptotes, then the transmutation curve becomes quite flat and nearly constant. What does this say about related area accumulation? How does one interpret the places where the transmutation curve crosses the x-axis in terms of area accumulation on the original curve?

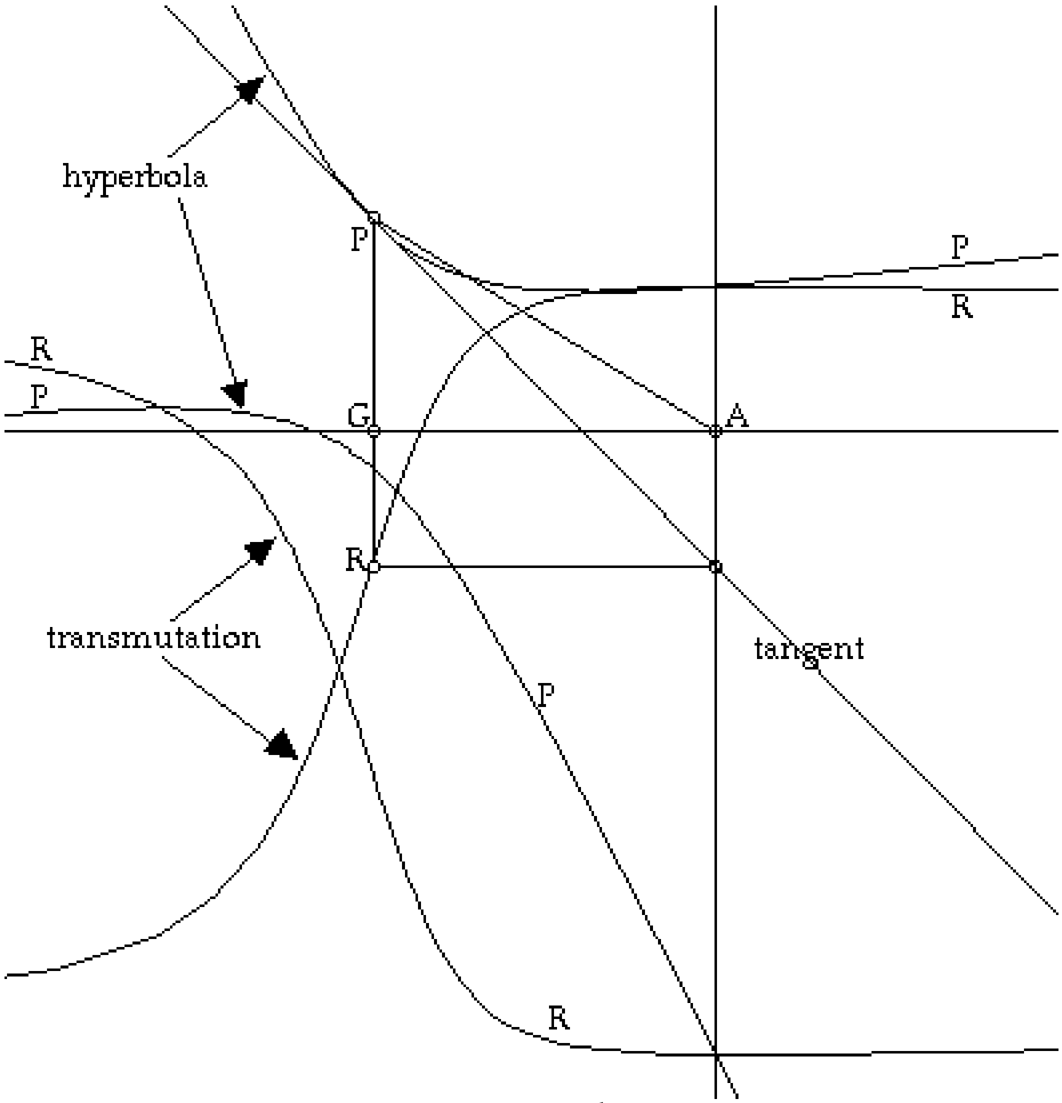

Figure2.14k

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http://www.quadrivium.info/mathhistory/CurveDrawingDevices.pdf

References can be found at
http://www.quadrivium.info/MathInt/Notes/MathIntRefsAlpha.html

Applets (animated versions of the illustrations) can be found at
http://www.quadrivium.info/MathInt/MathIntentions.html