1.2. Comparing areas

In this section you will learn how to compare the quantity of area for two different objects without using numbers or formulas. This gets at the meaning of what the term area means. Then there are two sections on geometry, since it will help to have lots of examples of geometric objects whose areas can be measured.

Contents

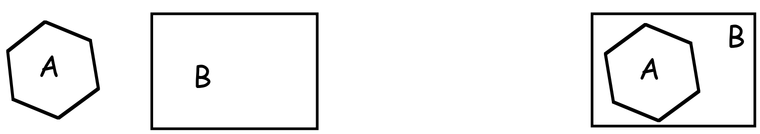
A. What is area? Rules for area

The *area* of a flat object or surface (such as a tabletop, the parking lot, or a sheet of paper) is the amount of space it covers. It also makes sense to consider the amount of space on a curved surface, such as an orange, a bagel, or the surface of a human body. Because you can say whether one object has a larger area than another, area is a quantity.

In the discussion that follows, A, B, C, etc. are objects that have area as one of their attributes, and area(A), read as "the area of A", is the quantity we are thinking about.

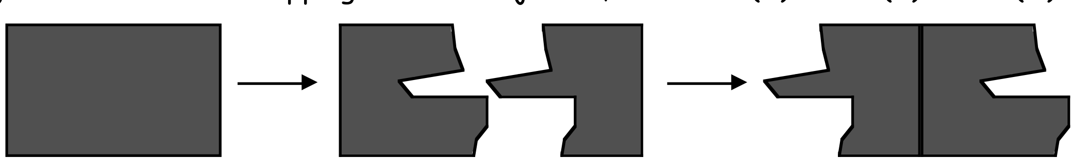
Here are some rules (axioms) for area:

1. If one object A can fit inside another object B, then area(A) < area(B).



2. If object A can be placed so that it exactly covers object B (that is, A is *congruent* to B), then area(A) = area(B).

3. If you cut an object A into pieces, say B and C, then area(A) = area(B) + area(C). Another way to think of this rule is that if you have two objects B and C, and put them together without overlapping to make object D, then area(B) + area(C) = area(D).



The third rule basically tells you how to add areas: put the areas together without overlapping. Notice that you can add areas without using numbers at all. Rule 3 also says that if you cut a shape up, then rearrange the pieces without overlapping them, the area stays the same.

B. Dissection

Dissection as a mathematical term means cutting up objects and rearranging the parts to make new objects.

Class activity 1. Comparing areas by dissection.

Materials:

Copies of shapes A through J printed on colored paper, a different color for each shape. There should be at one of each shape for each person, and some extras.

Scissors and tape.

Arrange the shapes A through J in order from smallest area to greatest area. No numbers are allowed, but feel free to use scissors and tape. Other copies, for reference, are on the attached pages. Don't cut these out.

Record results for the areas of the shapes for future reference.

Discussion question 1. Was it sometimes hard to tell whether two figures had the same area, or maybe just were really close? What could affect the accuracy of your measurements?

Can you identify any general methods for dissection, such as a way to make some kind of shape into rectangles?

Class activity 2. Liquid area? Dissection is sometimes an efficient way to compare areas, but sometimes it's just a pain in the neck. Wouldn't it be nice if you could take one amount of area and pour it into a different shape, keeping the same area?

Materials:

Dried beans, in several sizes (don't mix them)

Shapes for this activity (see end of this section). These are bigger versions of some of the shapes from Class activity 1. .

Figure out a way to compare areas with beans. Use the results of Class activity 1. , so that you already know how the areas compare. Does the size of beans you use make a difference? How?

C. Geometric terms and methods.

More definitions.

A *plane* is a mathematical object that is flat, has no thickness, has no edges, and goes on forever. Think of an infinite wall, floor, or tabletop. *Planar* means "in or on a plane".

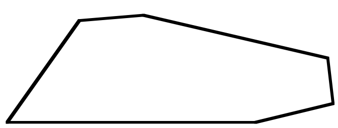
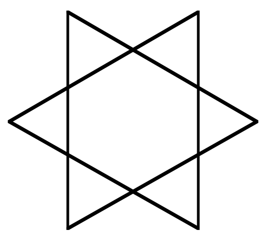
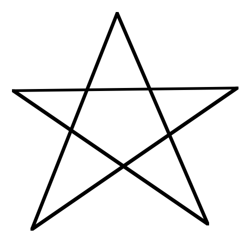
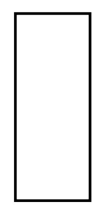
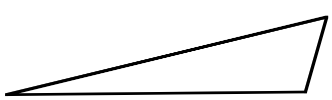
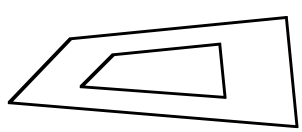
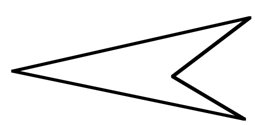
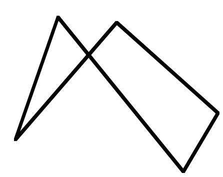
A *line* is a mathematical object that is straight, has no width or thickness, and goes on forever at both ends. A part of a line that has two endpoints in called a *line segment.*

A *polygon*[1] is a flat (planar) shape with straight (line segment) sides. If you walk along the edge of the polygon, then

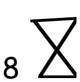
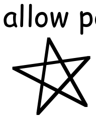
- i. you will eventually get back to your starting point, having covered all the sides once, without revisiting any other points[2]; and
- ii. there is an inside and an outside to the polygon. The inside should always stay on your same side (right or left, depending on which way you are walking.)

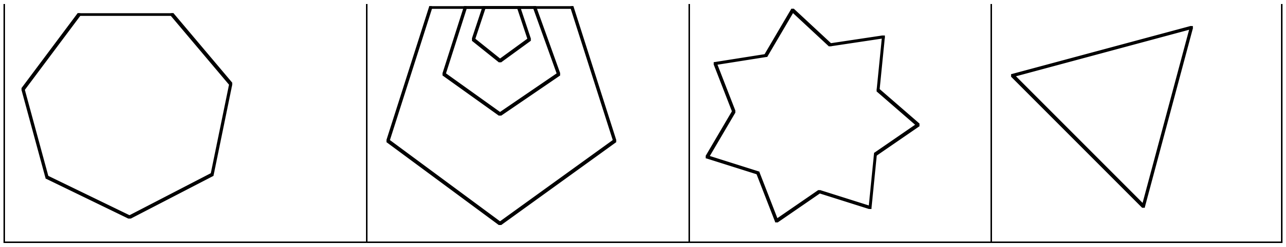
The sides of a polygon can also be called *edges*. The corners are called *vertices* (plural of vertex).

Which of these shapes are polygons? Why?

A 	B 	C 	D 
E 	F 	G 	H 
I	J	K	L

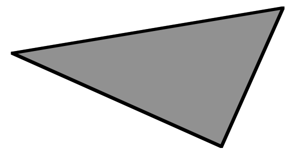
[1] *Poly* means "many" in Greek. *Gon* means "angle".

[2] Sometimes it's useful to allow polygons that cross over themselves, like a figure 8  or the kind of 5-pointed star  you draw to say, "Excellent work!" These are called complex polygons; the non-crossing kinds are then called simple polygons.

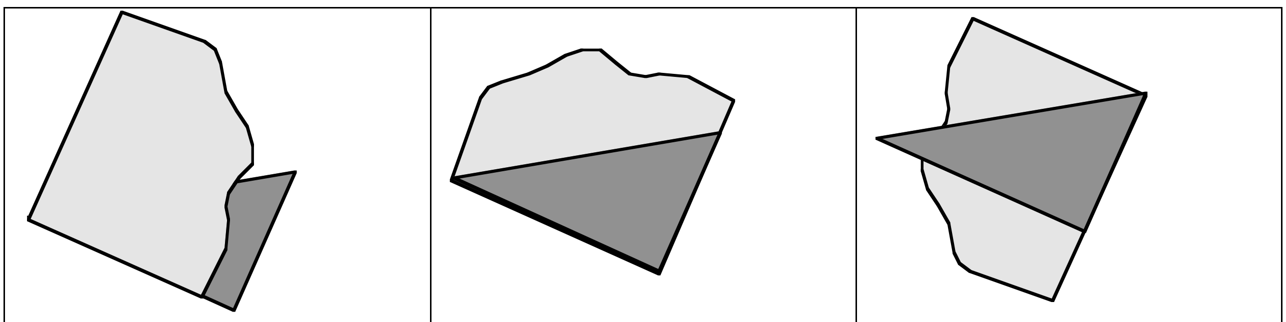


A *right angle* is a square corner. Two sides/lines/planes are called *perpendicular* if they meet at a right angle. A convenient way to compare or construct right angles on paper is to use the corner of another sheet of paper.

Example 1. Does this triangle have any right angles?



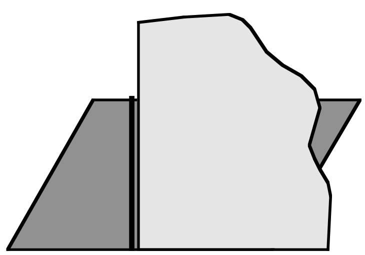
The lighter shaded object below represents the corner of a sheet of paper. It's not necessary to tear it off—just use any paper you have that's in good condition.



No	Yes	No

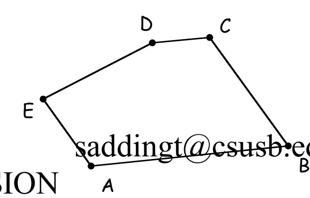
Example 2. Draw a line perpendicular to the bottom side of the parallelogram.

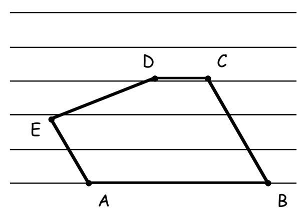
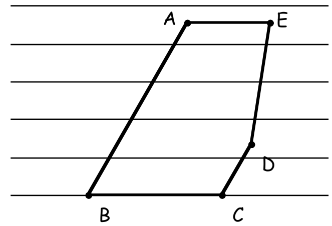
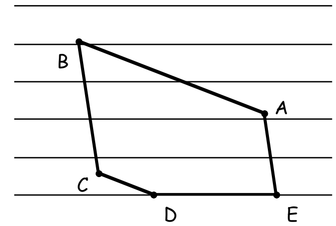
Line up the bottom of the parallelogram with one of the sides of the corner of the paper. Draw along the other side of the corner to get a line perpendicular to the base.



Lines (and segments) are *parallel* if they go in the same direction. Even for line segments that end, if they are extended, they will never intersect. A good example of parallel lines are the lines on notebook paper.

Example 3. Does this polygon have any pairs of parallel sides?



		
Yes, 1st pair: AB is parallel to CD.	Yes, 2nd pair: BC is parallel to EA.	No: DE is not parallel to any other side.

Hi! It's me, Etty Wanda.

1. My teacher, Mr. Groener, says that if two lines (infinite lines, not just segments; I asked) never intersect, then they are parallel. But I can see two lines in the room that aren't parallel, and will never intersect, even if you follow them forever.

2. Mr. Groener said that if a polygon has all right angles, it has to be a rectangle. I said that wasn't true, and then he sent me to the principal's office.

Four-sided polygons:

A *quadrilateral* is a 4-sided polygon.

A *rectangle*[3] is a polygon with 4 sides and 4 right angles.
A *square* is a polygon with 4 equal sides and 4 right angles.
A *parallelogram* is a quadrilateral with two pairs of opposite parallel sides.
A *trapezoid* is a quadrilateral with one pair of opposite parallel sides.

A *triangle* is a polygon with 3 sides.

Some types of transformations of objects:

To *translate* an object means to move it by sliding, without turning it or flipping it over.
To *rotate* an object means to turn it without flipping it over.
To *reflect* an object means to replace it by a mirror image. For objects on a flat surface, this can be done by flipping it over.

[3] "Right" comes from Latin "rect-", so rectangle literally means "right angle(s)", in this case 4 of them.

		
The dark R has been translated up and to the right.	The dark R has been rotated partway around.	The dark R has been reflected across the dotted mirror line.

Class activity 3. Acting out rotations and angles. A major use of angles is in measuring rotations. The amount of a rotation is often measured in parts of a turn.

a) Stand up and face the front of the classroom. Do a full turn. You should be facing the front again.

b) Again facing the front, do a half turn. Which way are you facing?

c) Again facing the front, do a quarter turn, either to the right or left. Point one arm in the direction you were facing originally, and the other arm in the direction you are facing now. Your arms form a right angle.

Class activity 4. Shapes with two triangles.

a) Make a triangle that has no right angles and 3 different side lengths.

b) Cut it out and trace it to make a second, congruent copy.

c) Make as many parallelograms as you can with the two triangles. (How many?)

d) Make as many other shapes as you can with the two triangles. How would you describe the shapes?

D. Tessellations: Math across the curriculum: Art[4].

A *tessellation* of the plane is a collection of shapes that cover the plane without gaps or overlaps. Here is one way to make a tessellation of the plane.

Class activity 5. Make a tessellation of the plane.

Materials: Index cards, carefully cut in half with a paper cutter. Tape, scissors.

a) Start with a rectangle. Draw a wiggly line that starts in the upper left corner and ends in the upper right corner. Cut along the line.

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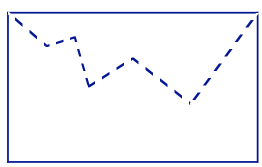
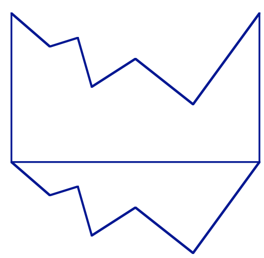
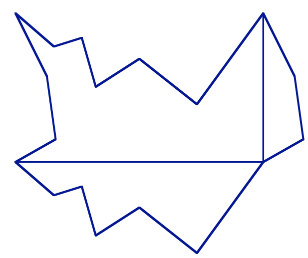
[4] M.C. Escher (1898–1972) was a graphic artist who used many mathematical ideas in his work. He is famous for tessellations made from the shapes of animals, people, and other recognizable objects. See the References section for art and teacher books on Escher and tessellations.

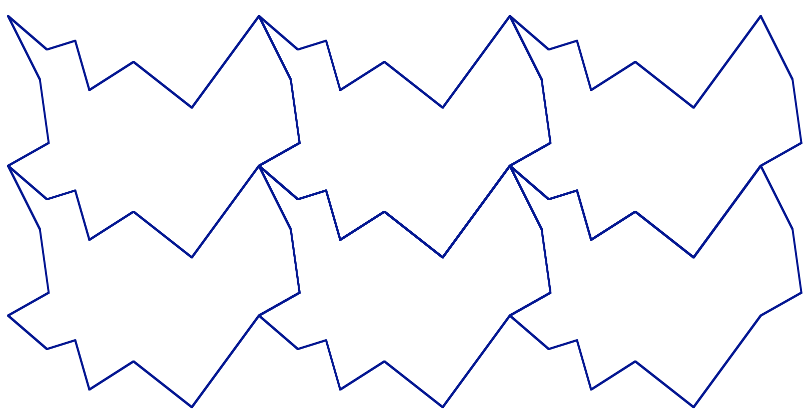
b) Tape the cut-out part to the bottom edge of the rectangle, matching straight sides.

c) Do the same with the left edge of the rectangle.

d) Trace the shape on a piece of paper. Slide (translate) it over so that one of the wiggly edges matches a traced edge, and trace again. Continue until you have covered the paper.

e) For an art project, use your imagination to make your shape into something: an animal, a face, a decorated shape. Do the same to each of the copies in your tessellation.

		
Step a)	Step b)	Step c)



Step d)

Question: How does the area of your tessellating piece compare with the area of the original rectangle?

Class activity 6. Saws and Ladders: A way to tessellate the plane with triangles.

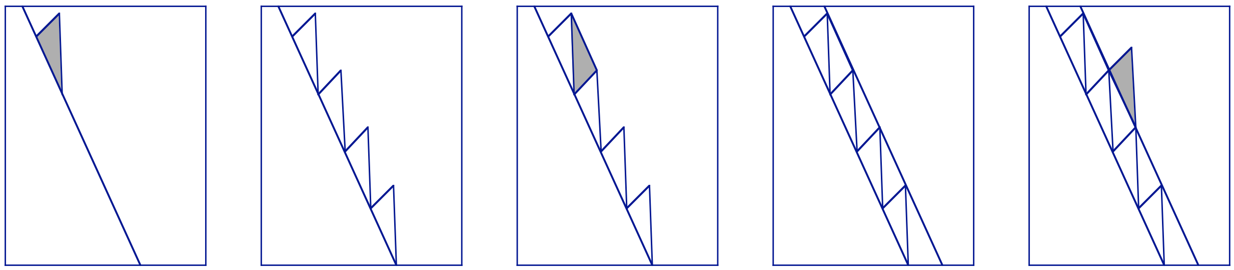
a) Start with a cut-out triangle. Get a blank sheet of paper. Using a ruler, draw a line from one edge of the paper to the opposite edge.

b) Put one edge of the triangle along the line. Trace the triangle.

c) Slide (translate) the triangle along the line to the end of the traced triangle; trace again. Keep going until you run off the paper.

d) Rotate the triangle a half-turn (don't flip over), fit it into a gap, trace again.

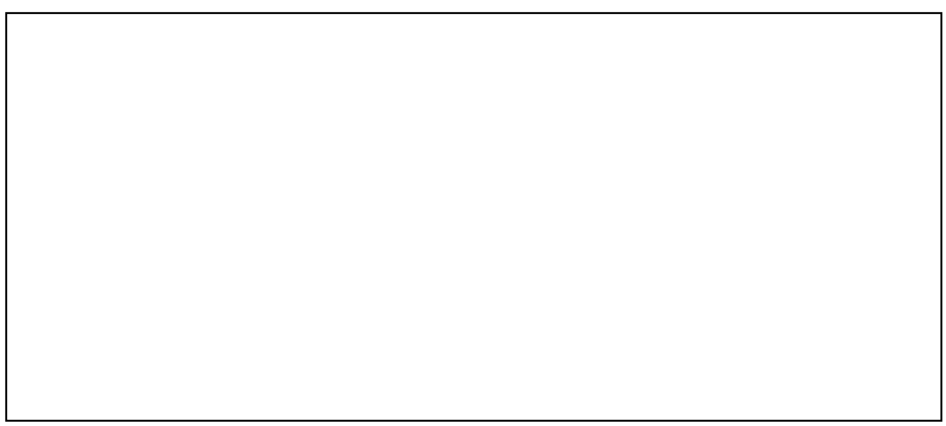
e) Keep going, forming new strips until you cover the whole paper.



E. Problems and exercises.

1. Make 5 different (that is, not congruent to each other) shapes that have the same area as the rectangle below, as follows. Draw or cut the shapes accurately.

   a. Make at least one other rectangle, at least one non-rectangular parallelogram, and at



   least one with curved sides.

   b. Can you make a triangle with the same area?

   c. For each shape, how do you know they have the same area? Explain with pictures or cutouts and some sentences. (No numbers!) For example, if you cut and reassemble your rectangle to make another shape, trace the steps on your paper to show what you did.

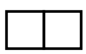
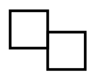
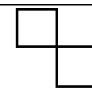
2. Finish the Saws and Ladders sheet you started in class. In this problem, two shapes that are congruent (same shape and size), but in different positions, are considered the same. When you have found a new shape, make a cutout from separate paper so

that you can see the shape clearly, outside of the confusion of the Saws and Ladders sheet. You can also move it around to check that any other shapes are different from that one.

a. Find as many different triangles as you can. Compare them to the triangle you started with: How are they different? What is alike about them?

b. Find as many different parallelograms as you can. Compare to each other: How are they different? What is alike about them? How are they related to the original triangle?

3. Square tile polygons. Get some square tiles, or cut them out of graph paper. You will be making shapes by joining the squares *edge to edge*. This means that, if two edges touch at more than one point, they touch for their entire lengths.

		
Two squares edge to edge	Not edge to edge	Not even a polygon

An edge-to-edge square tile polygon is called a domino: (The root "do-" means 2)

Again, in this problem, "different" means "not congruent to each other."

Find all the edge-to-edge square tile polygons that can be made with

a. This many tiles □ □ □. These are also called triominos. ("Tri-" means 3)

b. This many tiles □ □ □ □. These are also called tetrominos. ("Tetra" means 4 in Greek.) If you have ever played the computer game Tetris, you will recognize all the polygons so far.

c. This many tiles □ □ □ □ □. These are also called pentominos. ("Penta" means 5 in Greek.)

d. This many tiles □ □ □ □ □ □.

For each part, what is the same about the polygons you made? What is different?

4. Cut out copies of all the pentominoes you made in 3.c. Fit them together like a puzzle to make a rectangle. This can done in several ways, to make different sizes of rectangles. (There are a LOT of ways to do this. A quote from experts: "It is known that there are 2339 solutions for the 6 x 10 rectangle; 1010 solutions for the 5 x 12 rectangle; 368 solutions for the 4 x 15 rectangle; and 2 solutions for the 3 x 20 rectangle.")

5. Tessellations with other transformations.

a. Follow the instructions for tessellations in Class activity 5. , except flip one or both of your cut-off pieces before taping them. What do you have to do differently when moving and tracing the finished tile to cover the paper? Use the terms "translate", "rotate", and/or "reflect."

b. Start with a square instead of just a rectangle. Tape the part you cut out of the top to the right side instead of the bottom (with or without a flip), and tape the left side to the bottom. What do you have to do differently when moving and tracing finished tile to cover the paper? Use the terms "translate", "rotate", and/or "reflect." *Warning*: one of the rotate-and-flip combinations makes a tile that will *not* tile the plane. If you found this one, describe what you did, then try another.

c. Start with a number of squares, and make identical cuts on each. Tape them together in different ways, as described in parts a and b. Make tessellations out of each of the finished tiles, and compare the different patterns you get. How many different patterns are possible?

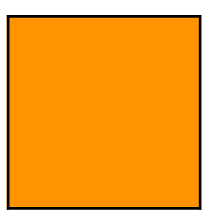
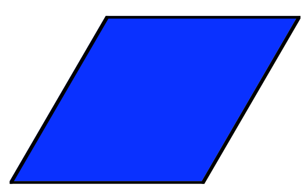
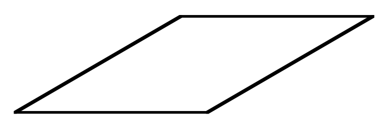
6. Use a tessellation created by the techniques in Class Activity 5 or Problem 5. Find a rectangular grid that the tessellation is based on as follows.

a. Choose a point on your tile, such as the tip of the fish's tail (if you have a fish.)

b. Find that point on each tile in the tessellation.

c. Use a ruler to connect the points to form a grid. Look for a pattern; don't just connect any points. You should find a grid formed by a rectangle the exact shape and size of the rectangle you started with before making your tiling piece.

7. Pattern blocks are a common math manipulative. They are plastic blocks in 6 shapes, in particular colors and sizes. Three of them are the square (orange), wide rhombus[5] (blue), and narrow rhombus (white). Use the cutouts on the attached sheets, color them, and show your work in steps by pasting them on your paper. Show the original shape with cut lines marked, then also show the dissected shape.

		
Square	Wide rhombus	Narrow rhombus

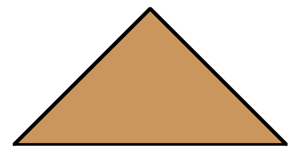
a. Use dissection to show that two narrow rhombi have the same area as the square.

b. Use dissection to show that the wide rhombus and the square have different areas. Which has a bigger area?

[5] A *rhombus* is a polygon with 4 equal sides.

c. How do the areas of the wide and the narrow rhombus compare (<, =, or >)? Show by dissection.

d. Here is a triangle that is not one of the pattern blocks. How does its area compare with the area of the narrow rhombus? Show or explain your reasoning.



General dissection methods, problems 8-12. In the diagrams, the back side of the polygon is shaded.

Show the steps with an example. Describe each step in words and trace the shapes for each step to show the process you used. Use the transformation words "translate", "rotate", and "reflect" (or flip) as appropriate. Labeling vertices will probably help keep track of things.

8. Parallelogram/rectangle

a. Find a way to dissect any parallelogram to a rectangle with as few steps as possible. (Hint: you will need to create a right angle with your cut.)

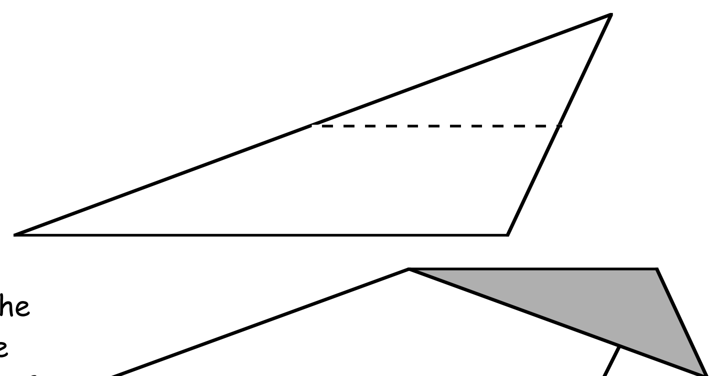
b. Find a way to dissect any rectangle to a parallelogram with as few steps as possible.

9. Triangle/parallelogram

a. Triangle to parallelogram.

i. Make a biggish triangle and cut it out.

ii. Make a fold parallel to one side (here the side is positioned at the bottom) halfway up the triangle, so that the top vertex ends up at the same level as the bottom of the triangle.

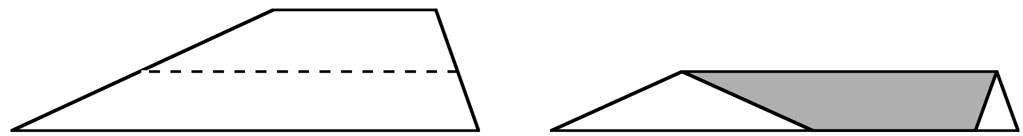


iii. Cut along the fold, and rearrange the pieces to make a parallelogram.

b. Parallelogram to triangle. Figure out a way to dissect any parallelogram to a triangle with one cut. Start with a new, different, parallelogram, but try to reverse the process in part a.

10. Trapezoid/parallelogram

a. Trapezoid to parallelogram. Fold a trapezoid so that the parallel sides are aligned. Cut along the fold line.



Rearrange the pieces to make a parallelogram.

b. Parallelogram to trapezoid. Figure out a way to dissect any parallelogram to a trapezoid with one cut. Start with a new, different, parallelogram, but try to reverse the process in part a.

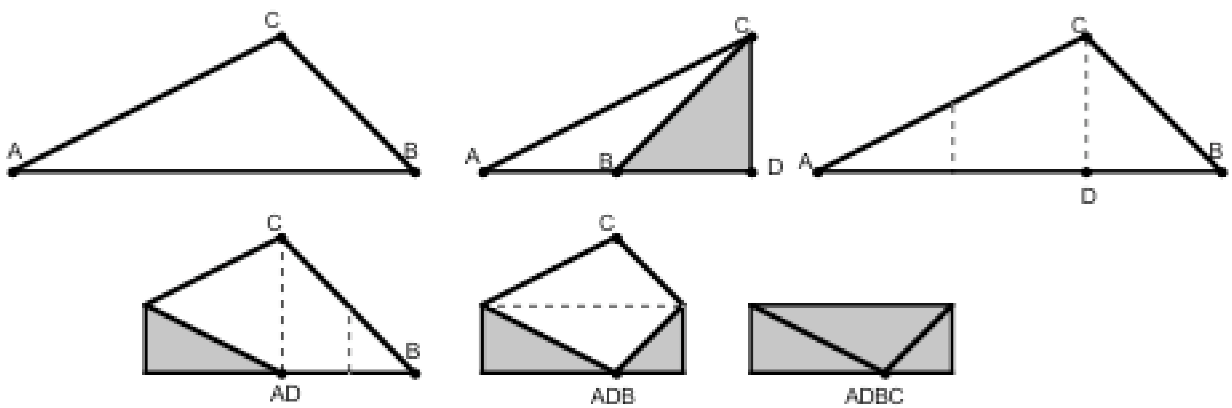
11. Folding a triangle

Using a straightedge, draw a triangle that takes up about half a piece of paper, and has three different angle measures. Cut it out, trace it, and fold as shown below.

Position the triangle so the long edge is at the bottom. Mark letters A, B, C.

Make a fold through C so that the long edge is folded along itself. Unfold.

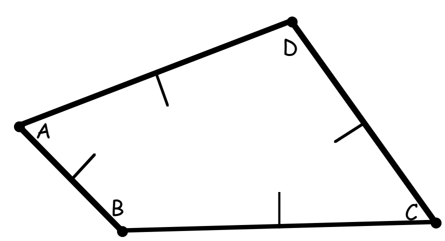
a. Fold A to D. Fold B to D. Fold C to D. (Note: you should have two layers of paper.)



What shape do you have? How does its area compare with the area of the triangle?

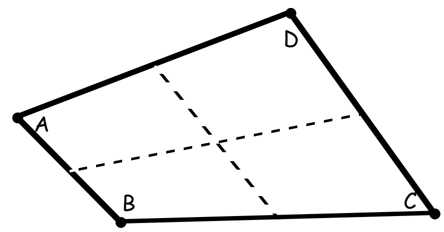
12. Quadrilateral to ?

a. Use a straightedge to draw a large quadrilateral. Try to make it look uneven–not a square, rectangle or parallelogram. Cut it out, color one side, and trace the shape for future reference. Mark the four corners A, B, C, D.

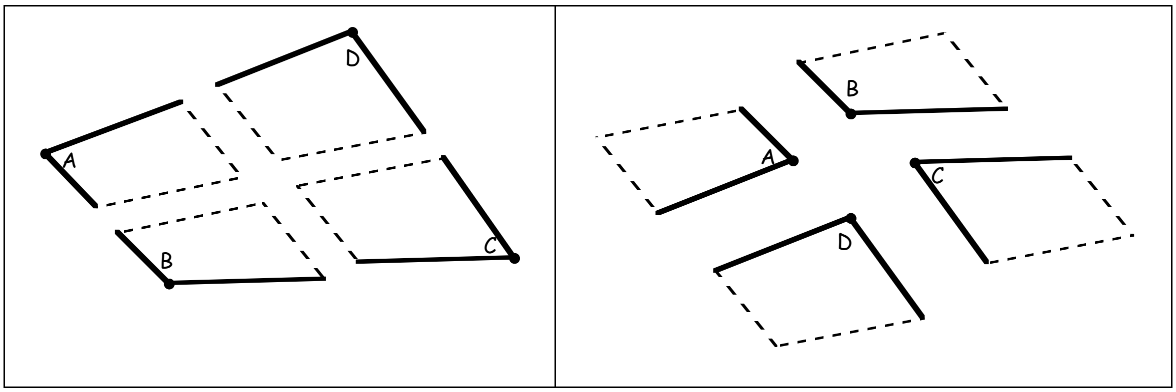


i. Fold each side in half and make a pinch to mark the midpoint. Don't fold all the way across, just the side.

ii. With a straightedge, draw lines connecting opposite midpoints, then cut along the lines to make four pieces, each with one of the corners A, B, C, D.



iii. Rearrange the pieces as shown, and fit them together. Don't flip them over. What specialized kind of shape do you get?



b. Give instructions how to move the pieces to rearrange them. Use the terms "translate" and "rotate", including what part of a turn to rotate.

c. (Geometry challenge.) Will this work for any quadrilateral? If not, give restrictions. Why does it work?

13. If you're thinking of an angle as a corner, a *straight angle* is a non-corner: walk down the street, get to the end of the block, and keep going straight. A straight angle is the same as a half turn if you're thinking of angles as amounts of rotations.

a. Explain how the construction in problem 11 gives evidence that the three angles of a triangle add to a straight angle.

b. Find several parts of your Saws and Ladders sheet that also give evidence that the three angles of a triangle add to a straight angle. Trace those parts on your paper.

14. On a circular clock (the kind with hands), how much time is:

a. A full turn of the second hand?

b. A full turn of the minute hand?

c. A full turn of the hour hand?

15. Look up figure skating moves (triple axel, Salkow, etc.) and classify them by number of turns.

F. References and Resources

Andrews, Angela, and Paul Trafton, *Little Kids, Powerful Problem Solvers*. Heinemann, 1999. Angela Andrews is a world-class kindergarten teacher, and the late Paul Trafton was a math education researcher who specialized in primary education. This book describes one lesson from Andrews' class for each month of the school year. Each lesson covers a different topic. She explains classroom management, and examines student thinking. See the lesson on triangles for a connection to some topics in this section.

Baggett, Patricia, and Andrej Ehrenfeucht, *Breaking Away from the Math Book.* A series of books of math and science activities that aren't texbook problems.

Frederickson, Greg N. 1997. Dissections: Plane & Fancy. Cambridge New York, NY, USA: Cambridge University Press.

Escher, M.C. Type "Escher" and/or "tessellations" into a web search engine and you will find lots of information. You can probably find books about his work in a library.

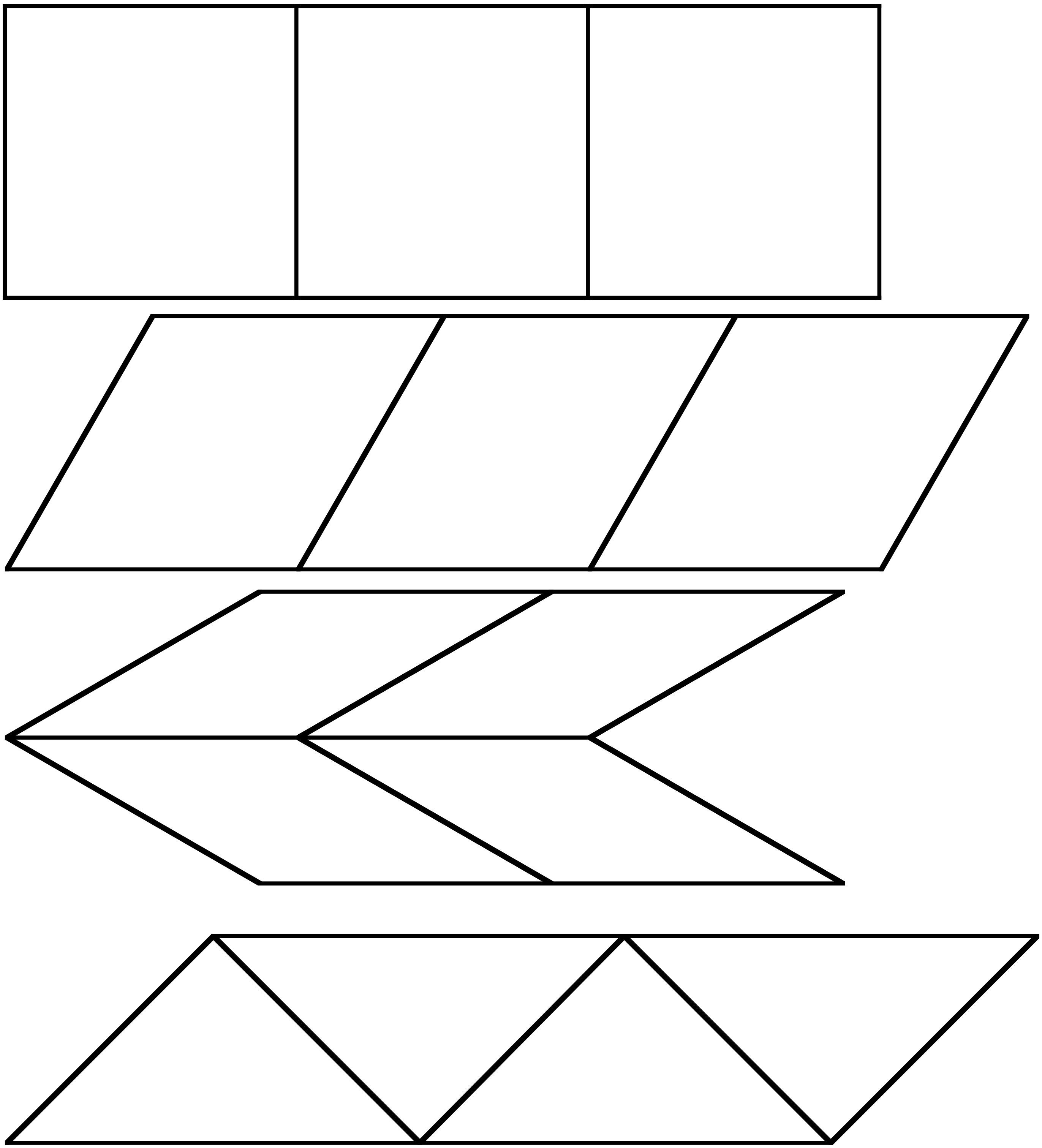
Puzzle Playground: http://www.puzzles.com/PuzzlePlayground/Dissections.htm An online collection of dissection puzzles.

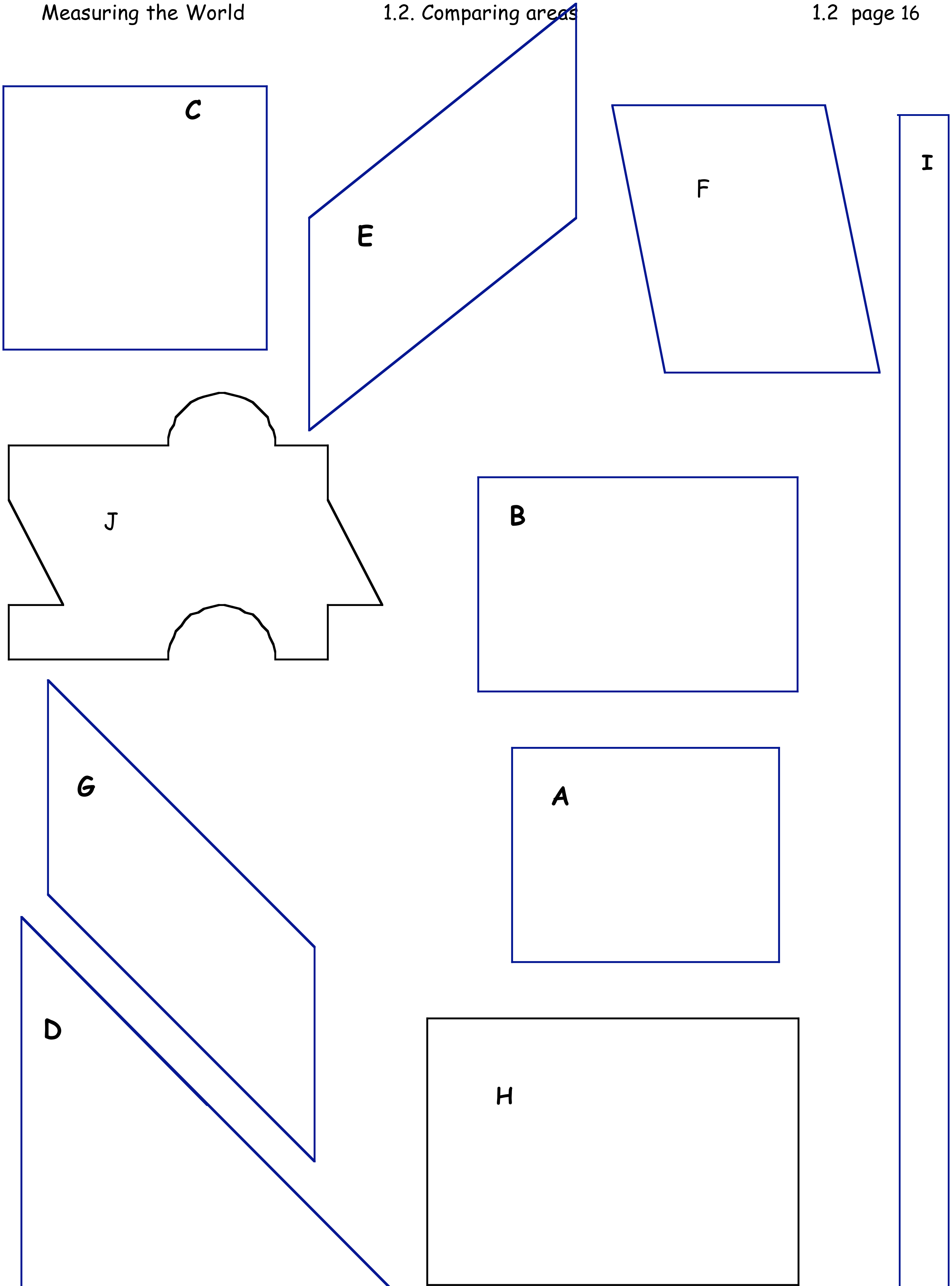
Schattschneider, Doris, *M.C. Escher: Visions of Symmetry*, 2nd edition. Harry N. Abrams, 2004. Gorgeous pictures, explanations of Escher's mathematical ideas.

Seymour, Dale and Jill Britton, Tessellations. Dale Seymour Publications, 1986. A book for teachers grades 6-12, full of class activities.

Yang, Tse-hsuan, Dissections under Dynamic Geometry. http://steiner.math.nthu.edu.tw/ne01/tjy/dissections/index.htm. A web site with animations of some very complicated dissections. Fun to watch! They are taken from the book by Frederickson, above (who must be a dissection fanatic.)

For problem 5: 3 pattern blocks, and another shape, twice normal size. Colors: square: orange; wide rhombus, blue; narrow rhombus, white; non-pattern block triangle, brown.





Pieces for Class Activity 1

Shapes for Class Activity 2.

Don't cut out.

Refer to area results
from Class activity 1.