Newton's Area Calculations

How did Newton know that he could create an area expression by summing up the area for each of the separate terms in a binomial expansion? He gave no reason at this point in the manuscript, but a reasonable reconstruction of thinking would most likely have been based on the area concepts of Calvalieri that are assumed in Wallis and in earlier manuscripts of Newton. Each individual power has its own characteristic ratio but a sum of different powers has no such constant ratio, hence the area contributed by each term in an expansion must be considered as a fraction of a separate rectangle in order to use the results about characteristic ratio. Consider the total area under the curve $y = ax^s + bx^t$ as the two separate pieces shown in Figure 7a, where the curve dividing the dark from the light area is $y = ax^s$.

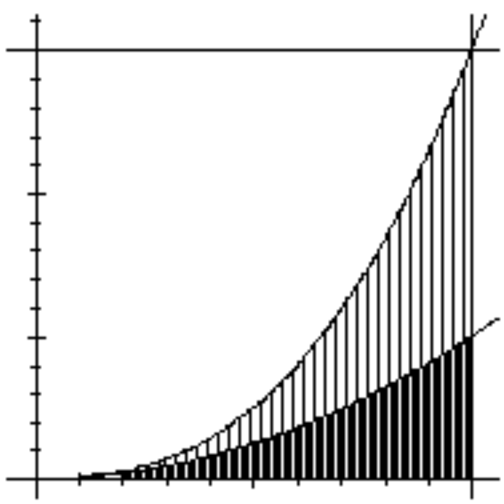

Fig. 7a

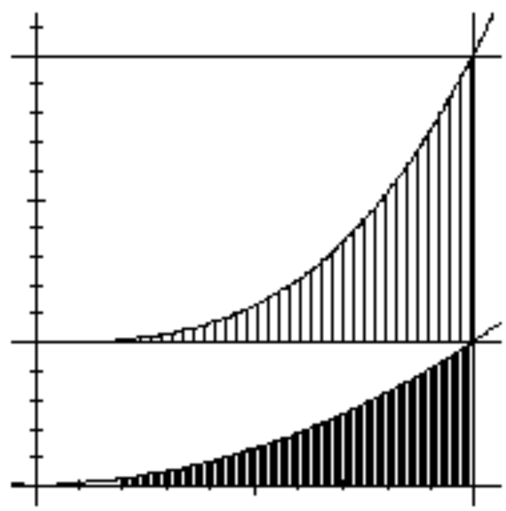

Fig. 8a

Leaving the darker area where it is, we could now move each of the line segments that compose the lighter area up to the line $y = k$, where k is the largest value of ax^s. (Think of moving the lighter area as if it were a deck of cards.) The lighter area will now fit inside a rectangle on top of the one that contains the darker area (see Figure 8a). The area of the bottom rectangle is ax^{s+1}, and the area of the top rectangle is bx^{t+1}. From Wallis we know that the dark area is $\frac{1}{s+1}$ of the bottom rectangle, and the lighter area is $\frac{1}{t+1}$ of the top rectangle, and hence the total area is $\frac{ax^{s+1}}{s+1} + \frac{bx^{t+1}}{t+1}$.

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