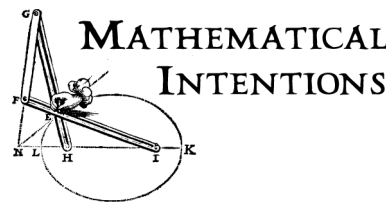


Euler and the Exponential Base e



In the next generation after Newton, Euler made extensive use of Newton's generalized binomial expansions greatly extending their range and utility. Newton used tables to construct infinite series, but once the method of formation of this series had been made clear Euler began using the series to construct tables. Euler conducted a lengthy series of investigations concerning the questions of which form of binomial expansions are most efficient for the construction of particular tables. From these investigations comes the modern notion of function and most of its attendant notation.

Euler's notations are familiar to us because during the eighteenth century he wrote a series of extremely influential textbooks which greatly standardized mathematical notation. Although the form and notation of Euler has been retained in our modern curriculum, much of the content and spirit of investigation has been lost. Euler believed strongly in empirical methods and this spirit pervades his famous precalculus text of 1748, the *Introductio in Analysin Infinitorum* (Euler, 1988). He felt strongly that the expansion of functions in infinite series is one of the basic tools of precalculus.

Binomial expansions expose many of the most important properties of functions as well as the connections between different functions. Euler was the first person to calculate the number e and show exactly how the hyperbolic area function is a logarithm. This is accomplished entirely using Newton's binomial expansions. Following this Euler extends the binomial series to complex numbers and expands the trigonometric functions. By looking at these series (what we now call the Taylor series for e^x , $\sin(x)$, and $\cos(x)$) Euler discusses the connections which allow him to see these functions as one family. The main content of Euler's work which is lost in our modern curriculum, is that by using empirical methods and binomial expansions all of these topics can be investigated at an elementary precalculus level.

Let us look at Euler's treatment of exponential functions. In Chapter VI of the *Introductio*, he presents the usual population and compound interest problems. However, he goes on in Chapter VII to derive several series for computing these functions. Consider the function a^x for $a > 1$. Since $a^0 = 1$, Euler lets $a^w = 1 + kw$, where " w is an infinitely small number." Here he is approximating a^x with a linear function on a small interval. k is the slope of the curve a^x at the point $(0,1)$. The value of the constant k depends on the base a . (For example, if $a = 10$, then $k = 2.30258\dots$) Now Euler expands $a^{wj} = (1 + kw)^j$, using the binomial theorem just as Newton would.

$$(20) \quad a^{wj} = (1 + kw)^j = 1 + \frac{j}{1}kw + \frac{j(j-1)}{1 \cdot 2}k^2w^2 + \frac{j(j-1)(j-2)}{1 \cdot 2 \cdot 3}k^3w^3 + \dots$$

Next he makes the substitution $x = wj$, or $j = \frac{x}{w}$, or $w = \frac{x}{j}$, noting here that since w is "infinitely small" we are now supposing that j is "infinitely large". (20) now becomes:

$$(21) \quad a^x = \left(1 + \frac{k}{j}x\right)^j = 1 + \frac{1}{1}kx + \frac{1 \cdot (j-1)}{1 \cdot 2j}k^2x^2 + \frac{1 \cdot (j-1)(j-2)}{1 \cdot (2j) \cdot (3j)}k^3x^3 + \dots$$

Now Euler points out that since j is infinitely large, $\frac{j-1}{j} = 1$, $\frac{j-2}{j} = 1$, $\frac{j-3}{j} = 1$, etc.

Hence $\frac{j-1}{2j} = \frac{1}{2}$, $\frac{j-2}{3j} = \frac{1}{3}$, etc. This conclusion is an intuitive use of limits that is quite similar to the way Wallis drew his conclusions about characteristic ratio. Now (21) becomes:

$$(22) \quad a^x = 1 + \frac{k}{1}x + \frac{k^2}{1 \cdot 2}x^2 + \frac{k^3}{1 \cdot 2 \cdot 3}x^3 + \frac{k^4}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \dots$$

Letting $x = 1$, (22) expresses the relationship between a (the base) and k (the slope).

$$(23) \quad a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \frac{k^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

Euler defines the number e as the base corresponding to the value of $k = 1$. (22) now becomes:

$$(24) \quad e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

Using (23) he computes the value of the base e .

$$e = 2.71828182845904523536028\dots$$

Looking back at the first part of (21) with $k = 1$, we see the familiar statement that $e^x = \left(1 + \frac{x}{j}\right)^j$

, or in modern terms $e^x = \lim_{j \rightarrow \infty} \left(1 + \frac{x}{j}\right)^j$.

Chapter VII of the *Introductio* also includes series that compute the inverses of (22) and (23). That is to say, logarithmic series are demonstrated as well as a direct method for computing k given the value of the base a . Many examples are shown, with much discussion of their relative efficiency for actual calculation.

Unlike many mathematicians, Euler never tried to mask the possible pitfalls of his methods. In Chapter VII he gives an example of how seemingly correct algebra can lead to paradoxical results. The contradiction arises because an alternating series for a particular number actually diverges. Euler's advice is to proceed and faith will return. That is to say his approach to mathematics was empirical. Like Newton, he created equations by analogy and then tested them in various ways for to see if the results were consistent. Formal proofs of the

conditions for the convergence of these binomial series were not given until nearly a century later by Gauss.

Throughout the *Introductio*, Euler made free use of complex numbers as well as the infinitely large and small quantities seen in the above example. He found that the series described above allowed him to extend the domain of e^x and $\ln(x)$ to the complex numbers. This ended a long controversy between Leibniz and Bernoulli concerning the appropriate definition of the natural logarithm of -1 (i.e. $\ln(-1) = i\pi$) (Cajori, 1913). In order to use binomial expansions to directly create series for $\sin(x)$ and $\cos(x)$, complex numbers are essential. The derivation begins by factoring the identity $\sin^2(x) + \cos^2(x) = 1$ into $[\sin(x) + i\cos(x)] \cdot [\sin(x) - i\cos(x)] = 1$ (see Euler, 1988, chapter XIII). Euler then goes on to display the profound connections between trigonometric and exponential functions. Since the complex exponential maps vertical lines onto circles centered at zero, it becomes natural to write trigonometric functions as linear combinations of exponentials, i.e.

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

Polar coordinates became very natural in this setting, and Euler makes extensive use of them in Book II of the *Introductio*.

An important point to consider here is that Wallis and Newton started with families of functions which through the use of extension, analogy, and interpolation gave rise to binomial series. Euler started with binomial series expansions, and by extension and analogy united exponential and trigonometric functions in one family. This beautiful circle of empirical investigations can be carried out at an elementary level and forms the grounded activity upon which first calculus and then differential equations were built.

References cited in the text can be found at
<http://www.quadrivium.info/MathInt/Notes/WallisNewtonRefs.pdf>